SPS Solution of the Week 10/17/2020-10/24/2020

Problem 1. Take the real plane $\mathbb{R}^2$ and remove from it a countable number of points $a_1, a_2, a_3 ... \in \mathbb{R}^2$. Show that given any two points of the remaining plane $\mathbb{R}^2 - \{a_n\}$, they can be connected by a continuous path.

A **continuous path** between two points a, b of a set M is a function $f : [0, 1] \to M$, such that f is continuous and $f(0) = a,\ f(1) = b$.

Hint: How many lines pass through a given point of $\mathbb{R}^2$?

Solution: The central idea is that *if we show there are an uncountable number of paths with disjoint images (except at the start and the end) then the removal of a countable number of points can only remove a countable number of these paths.*
Let the set of lines passing through a point $z \in \mathbb{R}^2$ be S_z. Suppose we are give 2 points x and y in $\mathbb{R}^2 - \{a_n\}$, S_x, S_y are both uncountable.
Given $\{a_n\}$, the set of lines passing through x (or y) which intersected at least one of these points is only a countable subset of S_x (or S_y).
Thus there are an infinite number of lines that pass through x (or y) that do not hit any point of $\{a_n\}$.
Pick such a line that passes through x, and another with a different slope at y. These lines are fully in $\mathbb{R}^2 - \{a_n\}$ and must intersect at some point due to unequal slope, the path defined by tracing from x to the intersection to y is one desired path. $\square$