SPS Problem of the Week 9/12/2020-9/19/2020

Problem 1. Given $\mathbb{R}^n$ and the standard basis vectors $\mathbf{e}_i$, we define a notion of multiplication by the following rules:

$$\mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_i = 2\delta_{ij}$$

Where δ_{ij} is the **Kronecker delta**.

(a) Notice that the space of matrices is quite similar to $\mathbb{R}^n$ and if we suppose that the $\mathbf{e}_i$s are matrices then we can identify our product $(\cdot)$ with matrix multiplication. Provide 3 matrices $\mathbf{e}_i$ which satisfy our requirements for $\mathbb{R}^3$.

(b) Assuming that realisations exist for $\mathbb{R}^n$. Prove that the $\mathbf{e}_i$ and their products give rise to a vector space of dimension 2^n.

*(c) Now for the real problem, suppose you have 2 vectors γ_1, γ_2 equal to some linear combination of $p-$vectors of $\mathbf{e}_i$. Explicitly, a $\mathbf{p-vector}$ of $\mathbf{e}_i$ is any product of the form $\mathbf{e}_{i_1}\mathbf{e}_{i_2}...\mathbf{e}_{i_p}$ where $(i_1, i_2, ..., i_p)$ is a permutation of a subset of $\{1, 2, ..., n\}$. For instance $\mathbf{e}_2\mathbf{e}_1\mathbf{e}_5$ is a $3-$vector in $\mathbb{R}^5$. $\gamma_1 \cdot \gamma_2$ gives rise to $p-$vectors, $(p-1)-$vectors,.. etc.
Determine an expression for obtaining only the $p-$vectors which occur in the product.