POTW Solution 8/22 - 8/29

Notice that if we can find a sequence of steps which change the orientation of only 1 key while leaving the others unchanged then we are done. (since we can then use the above sequence to manipulate each key individually).

Lets call our basic move which switches every key in the same row or column as the key (a,b) as $M_{a,b}$.

Out of thin air we claim that to switch the key (a,b) only, as prescribed earlier, you must make all the moves $M_{i,j}$ (in any order) where $(i,j) \in (\{a\} \times \{1,2,3,4\}) \cup (\{1,2,3,4\} \times \{b\})$

for $(a,b) = (2,3)$
The moves would be:
$M_{2,3}$; $M_{2,4}$; $M_{2,1}$; $M_{2,2}$; $M_{1,3}$; $M_{3,3}$; $M_{4,3}$

Now we provide some mathematical context to the problem so that ingenuity is no longer needed.

Consider any state of the board to be a vector in the space F_2^{16} ~~which~~ where F_2 is the binary field $(F,+,\times)$ with $F=\{0,1\}$.

Notice we can make our moves $M_{a,b}$ correspond to the state of the board when the move $M_{a,b}$ is applied to the board $(0,0,0\ldots,0)$ and that with this correspondance $M_{a,b}$ is a vector in F_2^{16}.

Observe that applying the Move $M_{a,b}$ to ~~th~~ a board in state $a \in F_2^{16}$ ~~is~~ results in a board with state $a + M_{a,b}$.

Now the question of whether we can go from any board $a \in F_2^{16}$ to a board $b \in F_2^{16}$ with moves $M_{a,b}$ is equivalent to the question of whether the vectors $M_{a,b}$ are a basis F_2^{16} and what we did earlier is to explicitly construct $b = a + \sum c_{a,b} M_{a,b}$ where $c_{a,b}$ are constants $\in F$.